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RAYLEIGH-TAYLOR STABILITY FOR A SHOCK WAVE-DENSITY DISCONTINUITY INTERACTION*

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ABSTRACT

Shells in inertial fusion targets are typically accelerated and decelerated by two or three shocks followed by continuous acceleration. The analytic solution for perturbation growth of a shock wave striking a density discontinuity in an inviscid fluid is investigated. The Laplace transform of the solution results in a functional equation, which has a simple solution for weak shock waves. The solution for strong shock waves may be given by a power series. It is assumed that the equation of state is given by a gamma law. The four independent parameters of the solution are the gamma values on each side of the material interface, the density ratio at the interface, and the shock strength. The asymptotic behavior (for large distances and times) of the perturbation velocity is given. For strong shocks the decay of the perturbation away from the interface is much weaker than the exponential decay of an incompressible fluid. The asymptotic value is given by a constant term and a number of slowly decaying discreet frequencies. The number of frequencies is roughly proportional to the logarithm of the density discontinuity divided by that of the shock strength. The asymptotic velocity at the interface is tabulated for representative values of the independent parameters. For weak

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shocks the sol ion is compared with results for an incompressible fluid. The range of density ratios with possible zero asymptotic velocities is given.

I. INTRODUCTION

A well known example of a fluid instability is the Rayleigh-Taylor type, caused by the acceleration of a heavy fluid by a lighter one. The growth rate of an interface perturbation is given in Chandrasekhar[1] for incompressible fluids. It is probably the effect which will limit the performance of inertial fusion pellets. A typical pellet design consists of one or two dense shells surrounded on each side by lower density material.[2,3] As each shell is first accelerated and then decelerated, one surface will be unstable. The initial acceleration is usually by shock waves. For example, when the outer shell accelerates the inner one, a shock wave is reflected back and forth between them. At each reflection a shock is transmitted into the dense material. It will eventually overtake an earlier transmitted shock. A rarefaction wave is returned, and this terminates the series of isolated impulsive shock accelerations with a continuous form of acceleration. A rarefaction may also be returned when a transmitted shock hits the outer edge of a shell. There are typically two or three shock accelerations of an interface before the series is ended. When the inner shell is decelerated, the situation is similar with the shock wave being reflected between the inner surface and the center of symmetry.

The purpose of this paper is to investigate the analytic properties of perturbation growth when a shock wave strikes a density jump. The initial conditions consist of two uniform materials with plane symmetry. A shock wave, whose direction is normal to the interface, is incident from one side. The interface is assumed to have a perturbation. The amplitude is small compared to the perturbation wavelength, so a linear analysis is adequate. There appears to be a simple analytical solution only for weak shocks (Section IV). The solution for strong shocks is given by a power series. The perturbation equations were solved numerically for three cases by Richtmyer (1960).[4]

The interface perturbation velocity for an incompressible fluid in the limit of impulsive acceleration is given by

$$w = kz_o v_d \frac{x - 1}{x + 1} \quad , \tag{1}$$

where k is the wave number of the perturbation, z_o, the initial amplitude, v_d, the velocity change of the interface, and x the density ratio. This gives the general scaling of the perturbation growth, and it is useful to compare the shock perturbation results with it. Away from the interface the perturbation decays as $\exp(-k|z|)$. It is a localized phenomenon, falling off to 1/500 one wavelength from the interface. The shock perturbation differs in that it falls off much more slowly, as $z^{-3/2}$ (Section IV). The shock interface perturbation velocity begins at zero and reaches an asymptotic value about the time a sound wave travels one wavelength of the perturbation. The asymptotic velocity is important as it determines the rate at which the materials mix with each other. For example, injection of high Z material into the deuterium-tritium fuel may quench the thermonuclear burn. A comprehensive survey of the asymptotic velocity was undertaken. A computer code was written which summed the power series solution and plotted the results. This was done for several thousand cases. About 200 of these are given in Figs. 2 and 3. An interesting question was for what parameters the velocity changed sign. This would mean there would be parameters where the velocity was zero or very small. The results show that this does not occur for density ratios greater than about 1.5 (Section V).

Heat flow and viscosity are neglected. In that case the perturbation scales with the wave number. Heat flow tends to damp out the perturbation for wavelengths less than some maximum value. This may be roughly calculated by comparing a characteristic damping time to a characteristic hydrodynamic time, the wavelength divided by the speed of sound. The use of viscosity and electron thermal conductivity of a plasma[5] give the maximum wavelengths for significant damping for viscosity and electron condition, respectively:

$$\lambda = 10^{-5}T^2\rho^{-1}Z^{-7/2}$$

$$\lambda = 10^{-3}T^2\rho^{-1}Z^{-1} \quad .$$

The wavelength is in cm, density, g cm^{-3}, and temperature in keV ($= 1.16 \times 10^7$ ^oK). The heat capacity of an ideal gas is assumed. During the implosion the temperature is less than a kilovolt, and wavelengths of interest greater than 10^{-3} cm, so damping is probably not significant. Damping by radiation is more complicated because of the wide variation of opacities. If we assume an optimum opacity, damping can occur at any wavelength. This is because the mass of hot and cold material scales with the wavelength, the same scaling as the hydrodynamic time. The optimum radiation mean free path is somewhat less than a wavelength. Each hot and cold region (one-half wavelength thick) receives roughly a black body flux = $\sigma T^4 = 10^{24}T^4$ erg-sec^{-1}-cm^{-2}, with a temperature characteristic of the other region. The result is damping if

$$T > 0.4\ \rho^{.4} \quad .$$

II. ZERO ORDER SOLUTIONS

We assume a γ law equation of state on each side of the density discontinuity. There is a simple scaling with the absolute pressure and density for these equations of state, and four independent parameters are left. These are the values of γ on each side of the density jump interface, the density ratio, x, and the shock strength. The initial pressure is p_a. The incident shock is from side b, with the pressure behind being p_b. The final pressure is p_c. The initial density on side b is ρ_{b2}. The densities on each side at the moment the shock strikes the density jump at the origin are ρ_{b1} and ρ_{a1}. The densities behind the reflected shock and the transmitted shock are, respectively, ρ_{b0} and ρ_{a0}. We have the initial density ratio,

$$x = \frac{\rho_{a1}}{\rho_{b2}} \quad .$$

The shock strength is characterized by ε,

$$p_a = p_b (1 - \varepsilon) \quad ,$$

$$0 < \varepsilon \leq 1 \quad . \qquad (2)$$

The four independent parameters are then γ_a, γ_b, x, and ε.

The conditions behind each shock wave are determined by the three equations conserving mass, momentum, and energy. The first two are

$$v_d = W \left(1 - \frac{\rho_f}{\rho_h}\right) \quad ,$$

and (3)

$$p_b - p_f = w_d \rho_f W \quad ,$$

where the material velocity behind the shock, w_d, and the shock velocity, W, are with respect to the material velocity ahead of the shock. The subscripts b and f stand for behind the shock and in front of the shock. For a γ law gas the conservation of energy gives

$$\frac{\rho_b}{\rho_f} = \frac{p_b + \mu^2 p_f}{p_f + \mu^2 p_b} \quad , \qquad (4)$$

with

$$\mu^2 = \frac{\gamma - 1}{\gamma + 1} \quad .$$

The equations for the reflected and transmitted shocks are solved simultaneously for the six unknowns: the two shock velocities, the two densities behind the shocks, the velocity of the interface, and, p_c, the pressure at the interface. These may be reduced to one equation for p_c:

$$p_c - p_a = (p_b - p_a) \frac{1 + A}{B + A} \quad ,$$

where (5)

$$A = \left[\frac{p_a + \mu_b^2 p_b}{p_c + \mu_a^2 p_b} \right]^{1/2} ,$$

$$B = \left[\frac{\rho_{b2}(1 - \mu_a^2)(p_b + \mu_b^2 p_a)}{\rho_{a1}(1 - \mu_b^2)(p_c + \mu_a^2 p_a)} \right]^{1/2} .$$

The incident shock may produce either a reflected shock or a rarefaction. It is clear that the condition for a reflected shock is $B(p_c = p_b) < 1$. For a weak shock (ϵ small) this reduces to

$$\frac{\rho_{b2}\gamma_b}{\rho_{a1}\gamma_a} < 1 \quad . \qquad (6)$$

For a strong shock ($\epsilon = 1$),

$$\frac{\rho_{b2}(\gamma_b + 1)}{\rho_{a1}(\gamma_a + 1)} < 1 \quad .$$

Equation (5) may be solved by iteration. For a weak shock

$$p_c = p_a(1 + a_1\epsilon + a_2\epsilon^2) \quad ,$$

$$a_1 = \frac{2\gamma_a}{\gamma_a + \gamma_b} \quad ,$$

(7)

$$a_2 = \frac{(a_1^3 - 2a_1)}{4(1 + \mu_b^2)} + \frac{(2a_1^2 - a_1^3)}{4(1 + \mu_a^2)} + a_1 \, ,$$

$$\tau = (\gamma\rho)^{1/2} \quad ,$$

where $1 < a_1 < 2$.

III. THE PERTURBATION EQUATIONS

The perturbation equations are simple. The complexity of the problem comes from the boundary conditions along moving boundaries. We assume an initial perturbation of the interface, $z_o \exp(ikx)$. The initial velocity parallel to the shock direction and initial pressure are zero in the first order of the perturbation.[4] As the solutions will show, the source of the first order perturbation is the initial perturbations of the shock fronts. Zero order variables behind each shock include

$$w_d = \text{material velocity} \quad ,$$

$$W = \text{shock velocity} \quad ,$$

$$\rho_o = \text{density} \quad , \qquad (8)$$

$$c = \text{sound speed} \quad ,$$

$$\alpha = \left(\frac{W - w_d}{c} \right)^2 \quad ,$$

$$\beta = 1 - \alpha \quad .$$

The last two are important for the perturbation solutions. The shock ju-- conditions may be used to give

$$\beta = \frac{\varepsilon}{1 + \mu^2} \quad , \tag{9}$$

where ε is the strength of each individual shock. It is determined by but not in general equal to the ε value for the incident shock, Eq. (2). Subscripts which indicate the particular shock are not used when the same analysis applies to both shocks. The velocities, w_d and W, are with respect to the material ahead of the shock. An inverted coordinate system is used for the reflected shock so that the velocities are positive. Because the initial perturbation of the shock strength is zero,[4] the shock front perturbations ($t = (^+)$) depend only on shock velocities. They are

$$z_{sa} = z_o \frac{W_o - W_a}{W_o} \quad ,$$

(1)

$$z_{sb} = (-)z_o \frac{W_o - w_{do} + W_b}{W_o} \quad ,$$

where the zero subscripts refer to the incident shock. The minus sign (-) for z_{sb} comes from use of the inverted coordinate system.

Perturbation variables are p_1, pressure, w_1, z-velocity, and u_1, x-velocity. The shock direction coordinate is z_1, and time is t_1. Perturbation equations are

$$\frac{\partial u_1}{\partial t_1} = - \frac{ikp_1}{\rho_o} ,$$

$$\frac{\partial w_1}{\partial t_1} = - \frac{\frac{\partial p_1}{\partial z_1}}{\rho_o} , \qquad (11)$$

$$\frac{\partial p_1}{\partial t_1} = -\gamma p_o \left(iku_1 + \frac{\partial w_1}{\partial z_1} \right) .$$

These are in a system co-moving with material behind the shock front. The pressure equation is for flow isentropic along a mass point. Entropy may vary for different mass points. It is determined by boundary conditions at the shock front. Entropy variation would enter into the pressure equation in the convective derivative (e.g., $w_1 \, \partial p_1/\partial z_1$), but this is a second order effect. It is unnecessary to include the density equation because it does not couple to the others.

It is convenient to go to scaled variables: $z = kz_1$, $t = kct_1$, $p = p_1/k\rho_o c$, $u = iu_1/k$, and $w = w_1/k$. Results are independent of wave number in the scaled variables. This shows that perturbation growth scales as the wave number. The scaled perturbation equations are

$$\frac{\partial u}{\partial t} = p \quad ,$$

$$\frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z} \quad , \qquad (12)$$

$$\frac{\partial p}{\partial t} = -u - \frac{\partial w}{\partial z} \quad .$$

The shock front position is $z_s = w_c t$, where $w_c = \alpha^{1/2}$.

The shock wave jump conditions may be differentiated to give velocity perturbations at the front in terms of the pressure perturbation.

$$W_1 = Ap_1 \quad ,$$

$$w_1 = Bp_1 \quad , \qquad (13)$$

where

$$A = \frac{1}{2W\rho_f(1 - \mu^2)} \quad , \quad \text{and}$$

$$B = \frac{\rho_b(1 - 2\mu^2) + \rho_f}{2\rho_f W(1 - \mu^2)} \quad .$$

The requirement that acceleration at the shock front be perpendicular to the front gives

$$u_1 = -ikw_d z_{1s} \quad ,$$

$$\frac{\partial u_1}{\partial t_1} = -ikw_d W_1 \quad , \tag{14}$$

where z_{1s} is the shock front perturbation. In the scaled variables

$$w = w_c^{-1} B_1 p \quad , \quad \text{and}$$

$$\frac{\partial u}{\partial t} = A_1 p \quad , \tag{15}$$

with

$$B_1 = BW\rho_f = 1 - \frac{\beta}{2(1 - \mu^2\beta)} \quad ,$$

$$A_1 = A\omega_0 w_d = \frac{\beta}{2\alpha} \quad .$$

The final form of A_1 and B_1 come from identities from the jump conditions, and reduce the boundary conditions to the two parameters, μ^2 and β.

IV. METHOD OF SOLUTION

Solutions are found by taking Laplace transforms in time. This results in two modes of solution (pressure modes) with

$$p(s,z) = a(s)\exp(\ell_2 z) + b(s)\exp(\ell_1 z) \quad ,$$

$$w(s,z) = -\ell_2 a(s)\exp(\ell_2 z)/s - \ell_1 b(s)\exp(\ell_1 z)/s \quad , \tag{16}$$

$$u(s,z) = a(s)\exp(\ell_2 z)/s + b(s)\exp(\ell_1 z)/s \quad ,$$

where

$$\ell_1 = -\sqrt{s^2 + 1} \quad ,$$

$$\ell_2 = +\sqrt{s^2 + 1} \quad ,$$

with the convention that $\ell_1 = -s$, s large and positive. There is a third non-compressional mode (zero pressure contribution) that is independent of time. This means it is important for asymptotic velocities (large t). The equation is

$$u + \frac{\partial w}{\partial z} = 0 \quad . \tag{17}$$

The Laplace transform in time along the shock front ($z_s = w_c t$) with transform variable r gives

$$w_3(r) = -w_c u_3(r)/r \quad . \tag{18}$$

The initial value of w_3 is put to zero. It may be shown that the solution is independent of this initial value.

A Laplace transform of all the modes must be taken along the shock front to satisfy the boundary conditions. The moving boundary results in each mode being defined for a different value of the transform variable. For example the pressure solution of the first mode is

$$p(z,t) = \int ds\, a(s) \exp(st + w_c \ell_2 t) \quad , \tag{19}$$

because $z = w_c t$ along the shock front. With $r = s + w_c \ell_2(s)$

$$p(z,t) = \int dr \frac{ds}{dr} \exp(rt) a(s) \tag{20}$$

and the transform of p is clearly $a(s_2)\, ds_2/dr$, where $r = s_2 + w_c \ell_2(s_2)$. The limits of integration of r must be the same as for s. This is satisfied because $w_c < 1$. Similarly the transform for the second mode is $b(s_1)\, ds_1/dr$ with $r = s_1 + w_c \ell_1(s_1)$. We also have

$$s_1 = \frac{r + w_c \sqrt{r^2 + \beta}}{\beta} \quad ,$$

and (21)

$$s_2 = \frac{r - w_c \sqrt{r^2 + \beta}}{\beta} \quad .$$

The boundary conditions for the transforms give

$$-\frac{\ell_2}{s_2} a^* - \frac{\ell_1}{s_1} b^* - \frac{w_c}{r} u_1(r) = w_c^{-1} B_1 (a^* + b^*) \quad ,$$

$$\frac{a^*}{s_2} + \frac{b^*}{s_1} + u_3(r) = \frac{A_1(a^* + b^*) + z_t}{r} \quad , \tag{22}$$

where

$$a^* = a(s_2)\,\frac{ds_2}{dr} \quad ,$$

$$b^* = b(s_1)\,\frac{ds_1}{dr} \quad ,$$

and $z_t = w_c z_{s1}(t = 0)$. Variables a and b may be solved in terms of u_3. The solution is simpler in terms of u_4:

$$u_3(r)(r^2 - \alpha) = [w_c A_3 u_4(r) + z_t/r]r^2 \quad , \tag{23}$$

where

$$A_3 = 2(A_1 - 1 + B_1)$$

$$= \frac{\beta^2(1 - \mu^2)}{\alpha(1 - \mu^2\beta)} \quad .$$

For variables $f = \ell_2 a$ and $g = \ell_1 b$,

$$f(s_2) = h_2(r)u_4(r) - w_c z_t/2r \quad ,$$

$$g(s_1) = h_1(r)u_4(r) - w_c z_t/2r \quad , \tag{24}$$

with

$$h_2(r) = (1 - B_1)r^2 - \frac{1}{2}\beta - \beta s_2 r \quad ,$$

$$h_1(r) = (1 - B_1)r^2 - \frac{1}{2}\beta - \beta s_1 r \quad . \tag{25}$$

Boundary conditions at the origin are continuity of velocity perpendicular to the interface and pressure for the solutions. Because the fluids are inviscid, they may slip with respect to one another parallel to the interface. It is necessary to use a common transform variable $s = s_a c_a = s_b c_b$. Remembering the inverted coordinate system for side b, we have

$$-\frac{f_a + g_a}{s} = \frac{f_b + g_b}{s} \quad ,$$

and (26)

$$\rho_{ao}\frac{f_a - g_a}{\ell_2(s_a)} = \rho_{bo}\frac{f_b - g_b}{\ell_2(s_b)} \quad ,$$

where

$$f(s_2) = g(s_1)\frac{h_2}{h_1} + \frac{1}{2} w_c z_t \frac{\frac{h_2}{h_1} - 1}{r} \qquad (27)$$

for each side. There are two equations with two unknowns, e.g., g_a and g_b. The difficulty is that each unknown is defined at two points. Because of the complicated way the points are related to each other, and the presence of variable coefficients, no direct analytic solution was found. However, there is one obvious method of solution. If g_i is known at $s_{1i}(s_{2i} = s/c_i)$, with

$$s_1 = [(1 + \alpha)s_2 + 2w_c \sqrt{s_2^2 + 1}]\beta^{-1} \quad , \qquad (28)$$

$i = a, b$, then $f_i(s_2)$ may be determined, and through them, g_a and g_b. It may be useful to use as an independent variable

$$y = s_1 + \sqrt{s_1^2 + 1} \quad , \qquad (29)$$

where $y(s_2) = y(s_1)C^{-1}$, $C = (1 + w_c)\alpha_o^{-1}$, and $\alpha_o = 1 - w_c$. We solve for g_a and g_b in a neighborhood of infinity in a power series in $1/s$. Repeated applications of Eq. (26) and (27) enlarge the area of solution closer to the origin. Near the origin the solution sweeps across the imaginary axis. It may be analytically continued across the axis. A (primary) singularity at s_1 will produce a (secondary) singularity at s_2. Any original singularity generates an infinite set of secondary singularities. It is simpler then to continue the solution only up to the imaginary axis. The solution in the left half plane may be continued up to the imaginary axis in the same way as in the right half plane. This produces a discontinuity in the solution on the imaginary axis near the origin. Primary singularities are either branch points (square roots) at

$s_i = i$ or poles where h_i has a zero. It turns out h_i has no zeroes when the solution is continued only to the imaginary axis. Let sides j and k be determined by $d = c_j/c_k > 1$. The secondary point of side j for $s_{1j} = i$ is off the Riemann plane and produces no secondary branch point. The secondary point for side k may lie on the imaginary axis, $c_k < |s| < c_j$, and a series of secondary branch points is generated. From Eq. (29) it is clear their positions are at

$$y_{kn} = i(d + \sqrt{d^2 - 1})\, C^{-n} \quad ,$$

$$n = 1, 2, \ldots \quad , \qquad (\ldots)$$

$$s_{kn} = \frac{1}{2}\, i(y_{kn} - y_{kn}^{-1}) \quad .$$

It is terminated by requiring that

$$|y_{kn}| > 1 \quad .$$

The domain $|y| < 1$ represents the continuation of the solution across the axis. All secondary points of $s_k = i$ are off the original plane. The branch points give the asymptotic behavior of the solution for large t and z; f_j has a branch point only at $s_j = i$; f_k has a branch point at $s_k = i$ and at the secondary points; g_k and g_j have branch points at the secondary points and at $s_i = i$, $i = j,k$. Except at the points where $\ell(s_i) = 0$, each branch point gives an asymptotic solution (large t) proportional to

$$\exp(i\omega t \pm \ell(s)z)t^{-3/2} \quad ,$$

$$s = i\omega \quad ,$$

$$\ell(s) = \sqrt{1 + s^2} \quad .$$

The asymptotic solution is qualitatively different for each side. On side j, the coefficient of z is negative real. On side k it is imaginary. On the side with greater sound speed (j) the solution decays exponentially, while it is oscillatory on side k. The asymptotic behavior due to the branch point at $s_1 = i$ is similar to that of

$$I = \oint ds \exp\left(st + \sqrt{1 + s^2}\, z\right) \quad .$$

This is integrated by changing the variable of integration to y and expanding the exponential. The integral is reduced to the residue at y = 0, giving

$$I = \frac{z}{\sqrt{t^2 - z^2}} J_1\left(\sqrt{t^2 - z^2}\right)$$

$$+ \sqrt{\frac{2}{\pi}} \frac{z}{(t^2 - z^2)^{3/4}} \cos\left(\sqrt{t^2 - z^2} - \frac{3}{4}\pi\right) \quad .$$

Equation (28) shows that s_1 tends to infinity for weak shock waves. With $g(s_1) = g_0 s_1^{-1}$ where $g_0 = O(z_t) = O(\beta)$, g is second order in β. As will be shown, h_2 is of order β^2 and h_1 of order one. Correct to a fourth order error in β,

$$f(s_2) = z_1(h_2(r)/h_1(r) - 1)\, r^{-1} \quad ,$$

and correct to a third order error

$$f(s_2) = -z_1 r^{-1} \quad , \text{ where } z_1 = \tfrac{1}{2} w_c z_t \; . \tag{31}$$

The value of $g(s_2)$ may be found from f_a and f_b, and this gives the weak shock solution. The velocity for the third mode is

$$w_3(r) = -\frac{w_c u_3}{r} = \frac{-w_c r}{r^2 - \alpha}\left(w_c A_3 u_4 + z_t r^{-1}\right)$$

$$= \frac{-\alpha A_3}{(r^2 - \alpha) h_1(r)} \left[r g(s_1) + z_1\right] - \frac{w_c z_t}{r^2 - \alpha} \quad . \tag{32}$$

With

$$s_1 = \left(r + w_c \sqrt{r^2 + \beta}\right) \beta^{-1} \quad ,$$

w_3 has a branch point at $r = i\beta^{1/2}$. On side k there may be additional branch points if g has singularities for $|s| > \beta^{-1/2}$. Correct to fourth order in r

$$w_3(r) = -\frac{\alpha A_3 z_1}{(r^2 - \alpha) h_1(r)} - \frac{w_c z_t}{r^2 - \alpha} \quad . \tag{33}$$

The contribution of the poles at $r = \pm w_c$ will be discussed later. With A_3 from Eq. (23) and $h_1(r) = -1/2\left(r + \sqrt{r^2 + \beta}\right)^2$ for small β, the discontinuity gives

$$w_3(t_s) = 2z_t(1 - \mu^2) \oint r \sqrt{r^2 + \beta} \exp (rt_s)\, dr$$

$$= 2z_t(1 - \mu^2)\, \beta^{3/2} \frac{\partial}{\partial x} (J_1(x)x^{-1}) \quad , \qquad (34)$$

with

$$x = \beta^{1/2} t_s = \beta^{1/2} w_c^{-1} z \quad .$$

For large z

$$w_3 \approx \exp (i\beta^{1/2} w_c^{-1} z)\, z^{-3/2} \quad .$$

The same asymptotic behavior occurs for strong shocks, except that on si[illegible] there may be additional frequencies. The solution differs significantly from the incompressible case where the perturbation is localized at the interface.

The recursion relation for the power series may be obtained by expansion in a power series in s_b:

$$V = f_a + g_a = -f_b - g_b = \sum v_\ell s_b^{-(2\ell+1)} \quad ,$$

$$P = s_a c_a \rho_{ao} (f_a - g_a)\, \ell_2^{-1}(s_a) = s_b c_b \rho_{bo} (f_b - g_b)\, \ell_2^{-1}(s_b) \qquad (35)$$

$$= \sum p_\ell s_b^{-(2\ell+1)} \quad .$$

We have

$$2f_b = P\ell_2\rho_{bo}{}^{-1}s^{-1} - V \;,$$

$$2g_b = -P\ell_2\rho_{bo}{}^{-1}s^{-1} - V \;,$$

and a similar relation for side a. Eq. (27) relates $f(s_2)$ and $g(s_1)$:

$$2h_1 f - 2h_2 g + 2z_1(h_1 - h_2)r^{-1} = 0 \;. \qquad (27a)$$

The initial expansion is in $y = s_1 + \sqrt{s_1{}^2 + 1}$.

$$P(s_1) = \frac{y^2 - 1}{y^2 + 1} \sum A_\ell y^{-2\ell-1} \;,$$

$$V(s_1) = \sum B_\ell\, y^{-2\ell-1} \;,$$

$$h_1(r) = -f_1 y^2 - f_2 + C^2 f_3 y^{-2} \;,$$

$$h_2(r) = f_3 y^2 - f_2 - C^2 f_1 y^{-2} \;, \qquad ([illegible])$$

$$f_1 = \beta^2\,[1 - 2\mu^2\alpha_o]/2(1 - \mu^2\beta) \;,$$

$$f_2 = \beta^2 \text{[illegible]}/4(1 - \mu^2\beta) \;,$$

$$f_3 = \alpha_o{}^4\,[2\mu^2(1 + w_c) - 1]/8(1 - \mu^2\beta) \;.$$

$P(s_2)$ and $V(s_2)$ are the same with $y \to yC^{-1}$. This appears to give a simpler recursion relation than others (and less of a truncation error problem for numerical calculations) because s is rational in y, $s_1 = 1/2(y - y^{-1})$. For side b Eq. (27a) gives

$$f_1(B_m - A_m) + f_2C^{-2}(B_{m-1} - A_{m-1}) - f_3C^{-2}(B_{m-2} - A_{m-2})$$

$$+ C^{-2m-1} [f_3(B_m + A_m) - f_2(B_{m-1} + A_{m-1}) \qquad (37)$$

$$- C^2f_1 (B_{m-2} + A_{m-2})]$$

$$-z_2(\delta_{mo}C^{-1} + \delta_{m1}C^{-2}) = 0 \quad ,$$

where

$$A_m = (2m + 1) \sum p_\ell \frac{2^{2\ell+1}(m + \ell)!}{(2\ell + 1)!(m - \ell)!} \quad ,$$

$$B_m = \rho_o c \sum v_\ell \frac{2^{2\ell+1}(m + \ell)!}{(2\ell)!(m - \ell)!} \quad ,$$

$$z_2 = \rho_o c z_t \alpha\alpha_o \quad .$$

For side a results are the same with the exception that the coefficient of z_2 is

$$(\delta_{mo}C^{-1}d_1^{-1} + \delta_{m1}C^{-2}d_1^{-3}) \quad ,$$

and

$$p_\ell \rightarrow p_\ell d_1{}^{2\ell-2m} \quad ,$$

$$v_\ell \rightarrow -v_\ell d_1{}^{2\ell-2m} \quad ,$$

$$d_1 = c_b c_a{}^{-1} \quad .$$

The singularities of the solution lie on $|y_j| = 1$, and so the power series using $A_m{}^j$ and $B_m{}^j$ converges everywhere in the complex plane except right along the discontinuity. Application of Eq. (26), typically once, but sometimes several times gives the solution on the discontinuity in terms of it off the discontinuity where the power series may be used.

V. RESULTS

Probably the most important parameter of the solution is the asymptotic velocity at the interface. This controls the rate at which the two materials mix with one another. The interface velocity is zero at $t = 0$, then may behave in a damped oscillatory fashion before settling down to its asymptotic value. The latter is determined by the pole at $s = 0$, Eq. (26), and we have

$$w_{as} = -f_a(s = 0) - g_a(s = 0) \quad . \tag{38}$$

In terms of the solution at $s_1(s_2 = 0) = 2w_c\beta^{-1}$,

$$w_{as} = 2(\rho_{oa}R_a \cdot \rho_{ob}R_b)/(\rho_{oa} + \rho_{ob})$$

where

$$R = \left(z_t - \frac{1}{2} A_3 g'(s_1)\right)/\left(2 + \frac{1}{2} A_3\right) \quad .$$

To a fourth order error

$$R = z_t/\left(2 + \frac{1}{2} A_3\right)$$

Away from the interface part of the solution behaves like exp(z) and part like exp(-z). The poles of the third mode ($r = \pm w_c$) cancel the positive exponential and add the same to the negative exponential. The net result is

$$w_{as}(z = 0) \exp(-z) \quad .$$

This is identical to the incompressible solution. The significant part of the asymptotic velocity away from the interface is then given by the third mode, discussed in Section IV.

It is convenient to use a normalized velocity,

$$u_{as} = w_{as}/w_{dd} z_0$$

to compare with the incompressible solution, Eq. (1). In the weak shock limit, we have an explicit solution. Using Eqs. (3), (7), and (10) gives

$$u_{as} = \frac{x - 1}{x + 1} + \epsilon F(x,y) \gamma_h^{-1} \qquad \text{([illegible])}$$

where

$$y = \left[\frac{\gamma_a p_{a1}}{\gamma_b p_{b2}} \right]^{1/2} \quad ,$$

$$F(x,y) = [(y-1)^2 + 4(x^2+y^2)y^{-1}(x+1)^{-1} - 2x - 2y]/(x+1)(y+1) \quad .$$

The condition for a shock is $y > 1$. F is plotted in Fig. 1. It is positive at $y = 1$, has a negative minimum, and then increases. Eq. (39) often gives a good approximation even for strong shocks. A rule of thumb (see Fig. 2 and 3) is that the second derivative of u_{as} with respect is ε is negative, so u_{as} is either close to Eq. (39) or somewhat smaller. An interesting question was whether u_{as} has negative values for a given density ratio. This may be approximately determined by comparing the absolute value of the minimum of $F(x,y)$ with $(x-1)(x+1)^{-1}$. The conclusion is that negative values are confined to $x < 1.5$. Figures 2 and 3 plot u_{as} as a function of shock strength, ε. Density ratios were $x = 1.25, 1.5, 2.0, 10.0, 100.0$. The density ratio for each curve may be picked off by its value at $\varepsilon = 0$. In Fig. 2, $\gamma_b = 1.5$. The values of γ_a (1.5, 2.0, 5.0) range from a soft equation of state to a very hard one. In Fig. 3, $\gamma_b = 2.0$. Results are similar to Fig. 2, except that the velocity for a strong shock ($\varepsilon \leq 1$) is usually larger. This is consistent with the linear approximation, Eq. (39). One curve is missing in Fig. 3 because the parameters give a rarefaction instead of a reflected shock wave.

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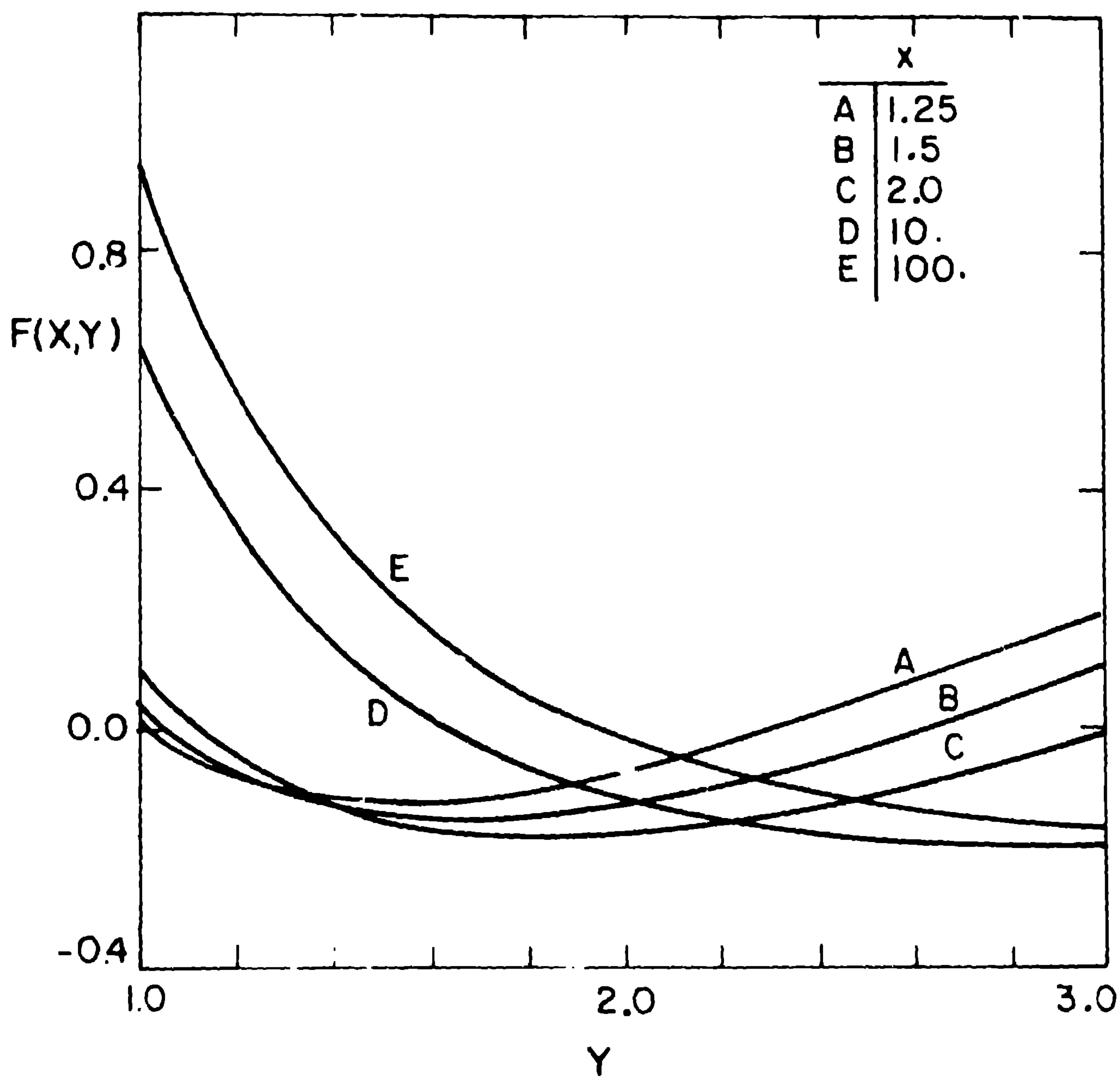
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B 1.5
C 2.0
D 10.
E 100.
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F(X,Y)
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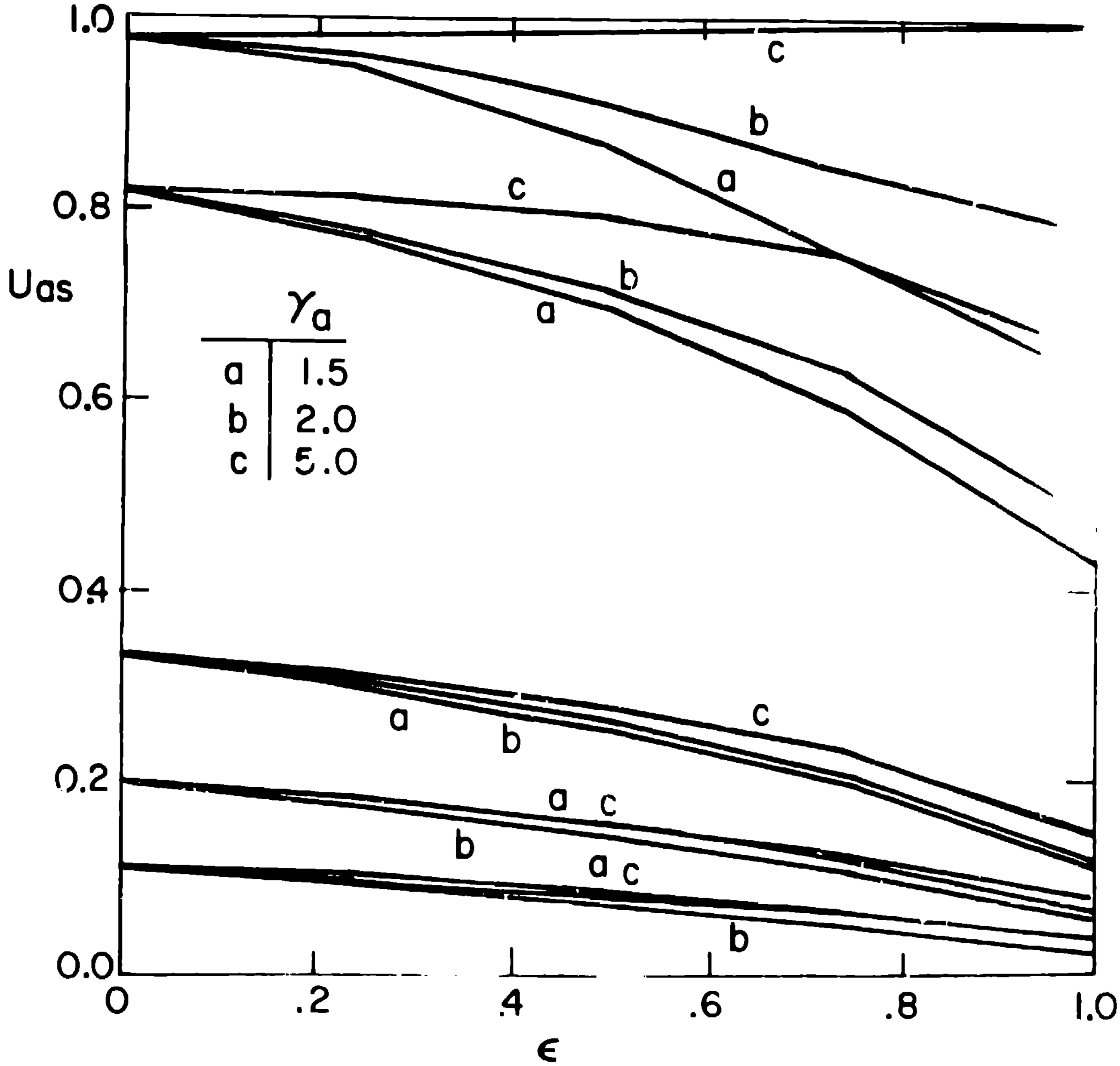
-0.4
E
D
A
B
C
1.0
2.0
3.0
Y



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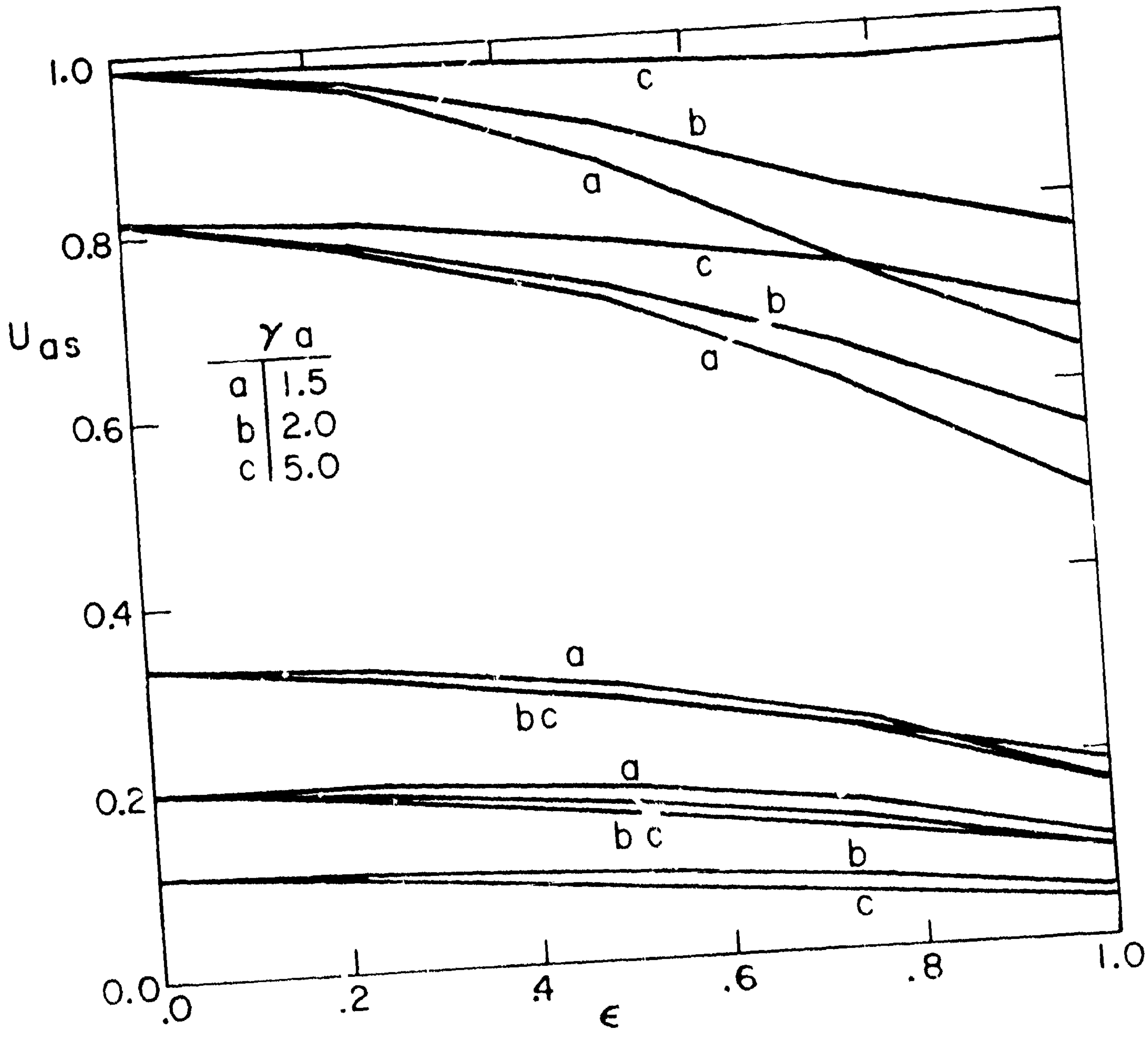
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